

A GENERALIZATION OF INVERSION FORMULAS OF PESTOV AND UHLMANN

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ABSTRACT. In this note, we give a generalization of the inversion formulas of Pestov-Uhlmann for the geodesic ray transform of functions and vector fields on simple 2-dimensional manifolds of constant curvature. The inversion formulas given here hold for 2-dimensional simple manifolds whose curvatures close to a constant.

1. INTRODUCTION

Let $(M, \partial M, g)$ be a C^∞ compact Riemannian manifolds with boundary. A variant of the classical Radon transform on Euclidean space is the *geodesic ray transform* on Riemannian manifolds defined as follows:

$$I_m f(\gamma) = \int_0^{l(\gamma)} \langle f(\gamma(t)), \dot{\gamma}^m(t) \rangle dt,$$

where $\gamma : [0, l(\gamma)] \rightarrow M$ is a maximal geodesic parameterized by arc length and m indicates the rank of the symmetric tensor field $f \in L^2(M)$.

We will be interested only in the cases $m = 0$ (functions) and $m = 1$ (vector fields) in this note and denote their geodesic ray transforms by I_0 and I_1 respectively.

The geodesic ray transform is not injective in general. One needs additional restrictions on the metric and one such restriction is to assume that the Riemannian manifold $(M, \partial M, g)$ is *simple* [Sha94] defined as follows:

Definition 1. *A compact Riemannian manifold with boundary is simple if*

- (a) *The boundary ∂M is strictly convex: $\langle \nabla_\xi \nu, \xi \rangle < 0$ for $\xi \in T_x(\partial M)$ where ν is the unit inward normal to the boundary.*
- (b) *The map $\exp_x : \exp_x^{-1} M \rightarrow M$ is a diffeomorphism for each $x \in M$.*

It is known that on a simple Riemannian manifold, $I_0 f$ uniquely determines f and $I_1 f$ uniquely determines the solenoidal component of f . For references to these works, we refer the book of Sharafutdinov [Sha94]. Then, similar to the classical Radon inversion formula, it is natural to ask whether there exists explicit inversion formulas for a function or a vector field in terms of its geodesic ray transform. In general this is a hard problem and such formulas are known only in special cases [Hel99].

In [PU04], Pestov and Uhlmann found Fredholm-type inversion formulas for the geodesic ray transform of functions and vector fields for simple 2-dimensional manifolds. These formulas become exact inversion formulas for 2-dimensional manifolds of constant curvature, even when conjugate points are present along geodesics.

A brief remark regarding notation. We use notation that is standard in integral geometry literature. Ours is consistent for the most part with [Sha94]. In this note, SM is the unit sphere bundle and $\tau(x, \xi)$ is the length of the maximal geodesic starting at $x \in \partial M$ in the direction $\xi \in \partial_+ SM := \{(x, \xi) \in \partial SM : \langle \nu(x), \xi \rangle \geq 0\}$.

The Fredholm-type inversion formulas of Pestov-Uhlmann are given by the following theorem:

Theorem 1. [PU04, Theorem 5.4] *Let (M, g) be a 2-dimensional Riemannian manifold. Then*

$$f + \mathcal{W}^2 f = \frac{1}{4\pi} \delta_\perp I_1^* (\alpha^* H(I_0 f)^- |_{\partial_+ SM}), \quad f \in L^2(M).$$

$$h + (\mathcal{W}^*)^2 h = \frac{1}{4\pi} I_0^* (\alpha^* H(I_1 \mathcal{H}_\perp h)^+ |_{\partial_+ SM}), \quad h \in H_0^1(M).$$

Here \mathcal{W} is the operator (\mathcal{W}^* is its L^2 adjoint) on $L^2(M)$ defined by

$$\mathcal{W}f(x) = \frac{1}{2\pi} \int_{S_x} \mathcal{H}_\perp \left(\int_0^{\tau(x, \xi)} f(\gamma_{x, \xi}(t)) dt \right) dS_x(\xi),$$

with

$$\mathcal{H}_\perp u(x, \xi) = \xi_\perp^i \left(\frac{\partial u}{\partial x^i} - \Gamma_{ij}^k \xi^j \frac{\partial u}{\partial \xi^k} \right).$$

As shown in [PU04], for manifolds of constant curvature, $\mathcal{W} = \mathcal{W}^* = 0$ and hence these formulas becomes exact inversion formulas.

In this note, we generalize these formulas to simple 2-dimensional manifolds whose curvatures are close to a constant. We show that in this case, the inversion formulas are given by convergent Neumann series expansions. Our main result is a generalization of the above result:

Theorem 2. *There exists a $C > 0$ such that if M is a simple 2-dimensional manifold with Gaussian curvature K such that $\|\nabla K\|_{C^0} \leq C$, the following inversion formulas hold:*

$$f = (I + \mathcal{W}^2)^{-1} \left(\frac{1}{4\pi} \delta_\perp I_1^* (\alpha^* H(I_0 f)^- |_{\partial_+ SM}) \right), \quad f \in L^2(M).$$

$$h = (I + (\mathcal{W}^*)^2)^{-1} \left(\frac{1}{4\pi} I_0^* (\alpha^* H(I_1 \mathcal{H}_\perp h)^+ |_{\partial_+ SM}) \right), \quad h \in H_0^1(M).$$

Remark: The proof relies on getting bounds for the operator \mathcal{W} (and hence \mathcal{W}^*) in terms of the gradient of the curvature K . Hence we will not give definitions of the terms appearing on the right hand side of the formulas in Theorems 1 or 2 which can be found in Pestov-Uhlmann's papers [PU05, PU04].

As shown in [PU04], \mathcal{W} is a smoothing integral operator extendible as a map $\mathcal{W} : L^2(M) \rightarrow C^\infty(M)$ with kernel,

$$W(x, y) = -Q(x, \exp_x^{-1}(y)) \frac{|\det(\exp_x^{-1})'(x, y)| \sqrt{g(x)}}{\sqrt{g(y)}}. \quad (1)$$

The function Q (equations (6) and (7)) and the partial differential operator ∂_θ (equation (4)) are defined in the appendix.

Acknowledgments: The author wishes to express his gratitude to Gunther Uhlmann and Leonid Pestov for their guidance and encouragement.

2. THE PROOF

We prove the following lemma. Here the derivatives are with respect to time. The functions a and b are defined in the appendix; see equation (5).

Lemma 1. *Let $\varphi = b\partial_\theta a - a\partial_\theta b$. Then denoting $K_\gamma = K \circ \gamma$, φ satisfies the following ordinary differential equation,*

$$\varphi^{(3)} + 4K_\gamma\varphi' + 2K_\gamma'\varphi = -2\partial_\theta K_\gamma,$$

with initial conditions,

$$\varphi(0) = \varphi'(0) = \varphi''(0) = 0.$$

Proof. First of all we have

$$ab' - a'b \equiv 1. \quad (2)$$

For, let $\phi = ab' - ba'$. Then $\phi' = ab'' - a''b$. From equation (5) we get that $\phi' = 0$ and so ϕ is a constant. Since $\phi(0) = 1$, we have the claim. With this we now show that $\varphi = b\partial_\theta a - a\partial_\theta b$ satisfies the ODE above.

$$\varphi' = b'\partial_\theta a + b\partial_\theta a' - a'\partial_\theta b - a\partial_\theta b'.$$

From (2) we get

$$b'\partial_\theta a + a\partial_\theta b' - b\partial_\theta a' - a'\partial_\theta b = 0.$$

This gives

$$\varphi' = 2(b'\partial_\theta a - a'\partial_\theta b) = 2(b\partial_\theta a' - a\partial_\theta b').$$

Differentiating again, we get

$$\varphi'' = 2(b''\partial_\theta a + b'\partial_\theta a' - a''\partial_\theta b - a'\partial_\theta b').$$

Using equation (5), this reduces to

$$\varphi'' = 2(-K_\gamma\varphi + b'\partial_\theta a' - a'\partial_\theta b'),$$

where $K_\gamma = K \circ \gamma$. Differentiating yet again, and as in the steps above, we finally get,

$$\varphi^{(3)} + 2K_\gamma'\varphi + 4K_\gamma\varphi' = -2\partial_\theta K_\gamma, \quad (3)$$

It now follows directly from these equations that $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$. \square

Notation: The norm $\|\cdot\|$ in the proof below denotes the sup norm unless indicated otherwise. Also in order to avoid proliferation of subscripts, we will use the same letter C to denote different constants.

Proof of Theorem 2. We now prove the main theorem. We rewrite equation (3) as a first order differential equation. We get

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \\ \varphi_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2K_\gamma' & -4K_\gamma & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2\partial_\theta K_\gamma \end{pmatrix}$$

where

$$\varphi_1 = \varphi, \varphi_2 = \varphi' \text{ and } \varphi_3 = \varphi''.$$

For simplicity, let us write this as a system of the form

$$X'(t) = A(t)X(t) + B(t),$$

where X, B and matrix A depend also on (x, ξ) . From [Cod61], since $X(0) = 0$, we have a solution of this differential equation to be

$$X(t) = \Phi(t) \int_0^t \Phi^{-1}(s) B(s) ds.$$

where Φ is the fundamental matrix of the homogeneous differential equation,

$$X'(t) = A(t)X(t).$$

Since the manifold is compact, we have $\|\Phi\|, \|\Phi^{-1}\| < \infty$. From the relation,

$$\partial_\theta K_\gamma = (\xi_\perp, \nabla K_\gamma),$$

and using the fact that SM is compact, we have a C such that

$$|\partial_\theta K_\gamma| \leq C \|\nabla K\|.$$

Combining these inequalities we get,

$$|\varphi'(x, \xi, t)| \leq |X(x, \xi, t)| \leq Ct \|\nabla K\|.$$

for some $C > 0$. Since

$$\varphi(t) = \int_0^t \varphi'(s) ds,$$

we have

$$|\varphi(t)| \leq Ct^2 \|\nabla K\|.$$

We can initially work with $\varphi''(t)$ and by the same arguments as above, we also get

$$|\varphi(t)| \leq Ct^3 \|\nabla K\|,$$

for a different constant C .

Since the manifold is simple, we have $b \neq 0$ for $t \neq 0$, since $b(0) = 0$. Now we write $b(t, x, y) = tb(x, y, t)$ with $\tilde{b} \neq 0$. Therefore for a suitable $C > 0$,

$$|q(x, \xi, t)| = |\partial_\theta \frac{a}{b}| \leq Ct \|\nabla K\|,$$

where the norm of ∇K is the sup norm. Since $tQ(x, t\xi) = q(x, \xi, t)$, we have

$$|Q(x, t\xi)| \leq C \|\nabla K\|.$$

Since the remaining terms in

$$W(x, y) = -Q(x, \exp_x^{-1}(y)) \frac{\det(\exp_x^{-1})'(x, y) \sqrt{g(x)}}{\sqrt{g(y)}}$$

are bounded above by compactness of M , we have

$$\|W\| \leq C \|\nabla K\|.$$

Therefore we have

$$\|\mathcal{W}\|_{L^2 \rightarrow C^\infty(M)} \leq C \|\nabla K\|.$$

So now choosing $\|\nabla K\|$ to be small enough, we have $\|\mathcal{W}\| < 1$. Hence we have inversion formulas involving Neumann series expansions recovering the function from its geodesic ray transform. A similar argument works for the recovery of the solenoidal part of a vector field from its geodesic ray transform. This completes the proof of the theorem. \square

APPENDIX A. THE KERNEL OF \mathcal{W}

For completeness and because the function q defined in equation (6) is critical for the proof of Theorem 2, we sketch below, Pestov-Uhlmann's [PU04] derivation of the integral kernel of the operator \mathcal{W} .

Recall that the operator \mathcal{W} is defined as

$$\mathcal{W}f(x) = \frac{1}{2\pi} \int_{S_x M} \mathcal{H}_\perp \int_0^{\tau(x,\xi)} f(\gamma(x,\xi,t)) dt dS_x,$$

where

$$\mathcal{H}_\perp = \xi_\perp^i \left(\frac{\partial}{\partial x^i} - \Gamma_{ij}^k \xi^j \frac{\partial}{\partial \xi^k} \right).$$

For a function u on SM , $\frac{\partial}{\partial \xi^k} u$ is defined as

$$\frac{\partial}{\partial \xi^k} u = \frac{\partial}{\partial \xi^k} (u \circ p)|_{|\xi|=1}, \text{ where } p(x, \xi) = (x, \xi/|\xi|).$$

We have

$$\mathcal{W}f(x) = \frac{1}{2\pi} \int_{S_x M} \int_0^{\tau(x,\xi)} \langle \nabla f, \mathcal{H}_\perp \gamma \rangle.$$

We define two Jacobi vector fields along the geodesic $\gamma(x, \xi, t)$ as follows: Let $x(s)$, $-\varepsilon < s < \varepsilon$ be a curve starting at x in the direction ξ_\perp . Now parallel translate the vector ξ along this curve, call it $\xi(s)$ and consider the variation by geodesics, $\gamma(x(s), \xi(s), t)$. The vector field

$$X(x, \xi, t) = \frac{d}{ds} \Big|_{s=0} \gamma(x(s), \xi(s), t),$$

is a Jacobi vector field along γ with the following initial conditions,

$$X(x, \xi, 0) = \xi_\perp, \quad D_t X(x, \xi, 0) = 0.$$

It can be also be written as

$$X(x, \xi, t) = \mathcal{H}_\perp \gamma(x, \xi, t).$$

We now define another Jacobi vector field by considering the variation by geodesics, $\gamma(x, \xi(s), t)$, where $\xi(s)$ is a smooth curve in $S_x M$ with initial tangent vector ξ_\perp . The Jacobi vector field

$$\partial_\theta \gamma(x, \xi, t) := Y(x, \xi, t) = \frac{d}{ds} \Big|_{s=0} \gamma(x, \xi(s), t) \quad (4)$$

has initial conditions,

$$Y(x, \xi, 0) = 0, \quad D_t Y(x, \xi, 0) = \xi_\perp.$$

Since X and Y are vector fields normal to γ and because of dimensional reasons these two fields must be proportional to the parallel translate of the vector ξ_\perp along the geodesic γ . Let this parallel translate be denoted $\dot{\gamma}_\perp$. Then there exists two smooth functions $a(x, \xi, t)$ and $b(x, \xi, t)$ such that

$$X = a \dot{\gamma}_\perp, \quad Y = b \dot{\gamma}_\perp.$$

The functions a and b satisfy the scalar Jacobi equations,

$$a'' + K a = b'' + K b = 0, \quad (5)$$

with

$$a(x, \xi, 0) = 1, \quad a'(x, \xi, 0) = 0, \quad b(x, \xi, 0) = 0, \quad b'(x, \xi, 0) = 1.$$

We now write,

$$\begin{aligned} \mathcal{W}f(x) &= \frac{1}{2\pi} \int_{S_x M} \int_0^{\tau(x, \xi)} \langle \nabla f, \mathcal{H}_\perp \gamma \rangle dt dS_x \\ &= \frac{1}{2\pi} \int_{S_x M} \int_0^{\tau(x, \xi)} \frac{a}{b} \langle \nabla f, Y \rangle dt dS_x \\ &= -\frac{1}{2\pi} \int_0^{\tau(x, \xi)} \int_{S_x M} \partial_\theta \left(\frac{a}{b} \right) f \circ \gamma dS_x dt. \end{aligned}$$

We now define a function q on

$$G = \{(x, \xi, t) : (x, \xi) \in SM, -\tau(x, -\xi) < t < \tau(x, \xi), t \neq 0\}$$

by

$$q(x, \xi, t) = \partial_\theta \left(\frac{a}{b} \right). \quad (6)$$

Using this we define a function $Q \in C^\infty(TM)$ by,

$$Q(x, t\xi) = tq(x, \xi, t). \quad (7)$$

The existence of this function Q follows from the fact that the geodesic $\gamma(x, \xi, t)$ is smooth as a function of $(x, t\xi)$ and the initial conditions of the Lemma 1. We now get the integral kernel $W(x, y)$ in equation (1) by a change of variables involving the inverse of the exponential map.

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